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A Nonlinear Gauge-Invariant Field Theory of Leptons

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A nonlinear, singularity-free, gauge-invariant field theory of leptons is proposed which incorporates the electron, muon, and tau. Fitting the known masses leads to a lepton radius of the order 10^{-16} cm, which is within the experimental range. The model considered suggests the possibility of a hierarchy of short-lived lepton states. Properties of the electron such as its energy-density distribution, Reissner–Nordström repulsion, and the fact that gravitation cannot play a significant role in its construction are discussed. All singularity-free charged particle models constructed from fields which were investigated approached the limit charge^2 = mass^2 (in geometrical units) as gravitation became a dominant force. It is suggested that this property may have great generality. The interaction-energy integrals which bind the particles are seen to increase as the energy increases, and it is suggested that a similar mechanism may also be responsible for quark confinement in hadrons.

1. INTRODUCTION

The essential goal of this paper is to propose a theory of leptons which precedes Dirac theory and electroweak theory in the sense that the particles are created from more fundamental constituents. This is achieved within the framework of a nonlinear, singularity-free field theory. The structures are described within a classical framework, with spherical symmetry, and it is assumed that quantization, as in the Dirac equation, would generate spin and magnetic moment for the particles. We have begun with leptons because these are the known "elementary" particles which can exist in isolation. A future goal would be one of extending the theory to quarks or to other constructs which might constitute hadrons.

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The Maxwell equations of classical electromagnetism lead to the description of an elementary charged particle as a singularity in the field. This presents the problem of an infinite self-energy, and its resolution via renormalization is not really satisfactory. Moreover, since the Maxwell equations are linear, equations of motion for the charge, as well as for the field, are required. As a result, classical electromagnetism is incomplete as a field theory. The concepts of particle and motion should be contained within the field theory, and not exist as independent entities (Einstein and Rosen, 1935). In a complete field theory, the particles and their motions (as well as the field dynamics) would be derived from variations of a Lagrangian constructed entirely from fields.

Early attempts to create an electron from the electromagnetic field succeeded only at the expense of "Poincaré stresses," which provide a phenomenological, rather than a fundamental, description of the particle. In an attempt to employ fields alone, Einstein (1919) showed that the addition of gravitation via general relativity still failed to create an electron. Born and Infeld (1934, 1935) developed an alternative theory of electromagnetism to that of Maxwell, which incorporated only the Maxwell tensor $F_{\mu\nu}$ in the Lagrangian, thus maintaining gauge invariance. While they were able to model particles, this work and later refinements (Hoffmann and Infeld, 1937) failed to eliminate singularities. It was recognized that the vector potential A_μ would have to appear explicitly in the Lagrangian to avert singularities. Such theories had been developed by Mie (1912, 1913) and others; while they were singularity-free, they were not gauge invariant.

Rosen (1939) (see Section 2) combined a scalar field with the electromagnetic field, thus maintaining gauge invariance. While this theory successfully modeled a charged particle without singularities, no particle of positive mass was found. Later, Rosen and Rosenstock (1952) showed that neutral particles of positive mass can be formed from a single scalar field and that the particle states are quantized (Section 2). It is natural to consider the addition of electromagnetism, and perhaps gravitation as well, to model elementary charged particles. In this manner one can determine the minimal modifications which will retain the simplicity and gauge invariance of Rosen (1939) in forming a charged particle entirely from fields while averting the problem of negative mass. It should be remarked that our present knowledge of the participation of the electron in weak-interaction processes lends further plausibility to the idea of describing it with the addition of a field other than electromagnetism and gravitation.

A variety of structures which have been considered by the present authors are described in this paper, and finally models which fit the known leptons well are presented. This in turn leads to a hierarchy of possible short-lived higher-mass lepton states. It is of interest that the experimentally determined lepton masses, in conjunction with our theory, lead to characteristic sizes for the leptons of the order 10^{-16} cm, which is within the present experimental limit.

In the course of the investigations, the role of gravitation was studied. First, it is easily seen that for the known charge and mass of an electron, gravitation is significant in the range $\lesssim 10^{-33}$ cm. However, every charged particle state which the authors were able to construct with gravitation playing a significant role led to a structure where the charge was of the order of or less than the mass in geometrical units, whereas $e/m \sim 10^{21}$ for an electron. If this result has general validity, then it suggests that the electron radius has a lower bound which is well above 10^{-33} cm, for otherwise particle states would exist with gravitation playing a significant role while e/m has this enormous value.

A variety of models was investigated in the limit as gravitation became a dominant force with the gravitational-field gradients reaching very large magnitudes. Of considerable interest is the fact that in every case the exterior Reissner–Nordström metric revealed that $e^2 \simeq m^2$, coming ever closer to equality as the model parameters were chosen to increase further the gravitational field intensities. In other words, the metric which is approached is never one with a so-called black-hole event horizon. This result could have important implications.

The paper is organized in the following manner. In Section 2 we review earlier theory, which forms the foundation of our present theory. The simplest gauge-invariant singularity-free particle model with negative mass (Rosen, 1939) is outlined, and new numerical data concerning the nature of the possible particle solutions are presented. The uncharged scalar-field model with quantized positive-mass particle states (Rosen and Rosenstock, 1952), which also plays a role in our new theory, is briefly reviewed.

In Section 3 the essential data concerning leptons are recalled. Arguments in favor of a finite size for leptons are given, and upper and lower bounds to their sizes are established, using experimental data and considerations involving gravitation. The picture of an electron as having a negative-energy inner core surrounded by a positive-mass outer layer, which emerges from the Reissner–Nordström metric in conjunction with the known values of electron mass and charge, is discussed. This leads to the conclusion that the Reissner–Nordström repulsion phenomenon (Papapetrou, 1974) is realized in an electron. The simplest conceivable models which might encompass the known lepton properties are also discussed briefly and are shown to be inadequate.

In Section 4 a successful theory incorporating three scalar fields is developed. Models of particles with quantized excitation states, which fit the known leptons well, are presented, and the energies of possible lepton states of higher excitation are given. The mechanism of confinement (which might also describe quark confinement in hadrons) is discussed.

The role of gravitation is developed in Section 5. The simplest particle model within the present framework is analyzed, and a detailed example is given. The result, that $e^2 \simeq m^2$ as the effect of gravitation becomes dominant, is described, and it is noted that this appears to have a considerable generality. The reasons for concluding that gravitation does not play a significant role in the structure of elementary particles with which we are familiar are given.

A concluding discussion of perspectives on the particle problem is presented in Section 6.

2. FOUNDATIONS

To avoid singularities in a theory constructed from the electromagnetic field, the four-vector potential A_μ as well as the Maxwell tensor $F_{\mu\nu}$ must appear in the Lagrangian. This raises the problem of maintaining gauge invariance, i.e., the invariance of the theory under a gauge transformation $A_\mu \to A'_\mu = A_\mu + \lambda_{,\mu}$, where λ is an arbitrary function of x^μ and a comma denotes a partial derivative. This problem is solved (Rosen, 1939) by introducing a complex scalar field ψ which, under a gauge transformation for A_μ, simultaneously undergoes a local phase transformation (rotation) $\psi \to \psi' = \psi e^{i\varepsilon\lambda}$ (ε = real constant). Then, in addition to the simple scalar $\psi\bar{\psi}$ (bar = complex conjugate), one can introduce derivatives of ψ which maintain gauge invariance provided they appear in the form $D_\mu\psi$, $D_\mu \equiv \partial_\mu - i\varepsilon A_\mu$. This is because $D_\mu\psi \to D'_\mu\psi' = e^{i\varepsilon\lambda}D_\mu\psi$ under the local phase transformation above, and hence quantities such as $(D^\mu\psi)\overline{(D_\mu\psi)}$ are both scalars and gauge invariant. In present-day parlance, D_μ is referred to as the "gauge-covariant derivative" (Quigg, 1983) and it arises in quantum mechanics as the operator which maintains invariance under local phase rotations.

In Rosen (1939) the simplest scalars $F_{\mu\nu}F^{\mu\nu}$, $(D^\mu\psi)\overline{(D_\mu\psi)}$, and $\psi\bar{\psi}$ were combined to form the Lagrangian

$$L = -F_{\mu\nu}F^{\mu\nu}/8\pi - (D^\mu\psi)\overline{(D_\mu\psi)} + \sigma^2\psi\bar{\psi} \qquad (\sigma = \text{real constant}) \quad (2.1)$$

which, after variation with respect to A_μ and $\bar{\psi}$, respectively, yield the field equations

$$F^{\mu\nu}_{;\nu} = -4\pi J^\mu \quad (2.2)$$

$$J^\mu = (1/2)i\varepsilon[\bar{\psi}D^\mu\psi - \psi\overline{(D^\mu\psi)}] \quad (2.3)$$

$$(D^\nu\psi)_{;\nu} - i\varepsilon A_\nu(D^\nu\psi) + \sigma^2\psi = 0 \quad (2.4)$$

where a semicolon denotes the usual covariant derivative. Thus, the electromagnetic and scalar fields, via the nonlinearities of the theory, create their own current sources J^{μ} for a complete self-contained field theory.

The energy-momentum tensor is

$$\begin{aligned} T_{\mu\nu} = {} & (1/16\pi)(g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - 4F_{\mu\alpha}F^{\alpha}_{\nu}) \\ & + (1/2)[g_{\mu\nu}\{(D^{\alpha}\psi)\overline{(D_{\alpha}\psi)} - \sigma^2\psi\bar{\psi}\} \\ & - D_{\mu}\psi\overline{(D_{\nu}\psi)} - \overline{(D_{\mu}\psi)}(D_{\nu}\psi)] \end{aligned} \tag{2.5}$$

The electromagnetic part of T_{00} is positive semidefinite, while the remainder (the "matter part") is negative semidefinite.

For solutions in which the charge is static and spherically symmetric, one can choose a gauge for which the four-vector potential is

$$A^0 = \varphi(r), \qquad A^k = 0, \qquad k = 1, 2, 3 \tag{2.6}$$

The four-current is

$$J^0 = \rho(r), \qquad J^k = 0$$

and the scalar field is most generally expressed in the form

$$\psi = \theta(r)e^{-i\varepsilon\mu t} \tag{2.7}$$

where θ is a real function of r, μ is a real constant, and $r = (x^2 + y^2 + z^2)^{1/2}$. In terms of these quantities, the field equations (2.2)-(2.4) are

$$\nabla^2\theta + [\varepsilon^2(\varphi + \mu)^2 - \sigma^2]\theta = 0 \tag{2.8}$$

$$\nabla^2\varphi + 4\pi\rho = 0 \tag{2.9}$$

$$\rho = \varepsilon^2(\varphi + \mu)\theta^2$$

with boundary conditions $d\theta/dr = d\varphi/dr = 0$ at $r = 0$ for regularity at the origin, $\theta \to 0$ exponentially as $r \to \infty$ for particlelike structure, and $\varphi \sim \text{const}/r$ as $r \to \infty$ for asymptotic Coulomb behavior of the electrostatic field.

The equations are easiest to examine in terms of dimensionless variables x, y, z via the substitutions

$$\theta = \sigma y/(4\pi)^{1/2}\varepsilon, \qquad \varphi + \mu = \sigma z/\varepsilon, \qquad r = x/\sigma \tag{2.10}$$

in terms of which they are

$$d^2(xy)/dx^2 = x(1 - z^2)y, \qquad d^2(xz)/dx^2 = -xy^2z \tag{2.11}$$

The total charge is

$$e = 4\pi \int_0^{\infty} \rho r^2\, dr = \alpha/\varepsilon, \qquad \alpha \equiv \int_0^{\infty} y^2 z x^2\, dx \tag{2.12}$$

and the energy density is

$$T_{00} = (1/8\pi)(d\varphi/dr)^2 - (1/2)\varepsilon^2(\varphi+\mu)^2\theta^2 - (1/2)(d\theta/dr)^2 - (1/2)\sigma^2\theta^2 \tag{2.13}$$

After an integration by parts, the total energy $W = 4\pi \int_0^\infty T_{00} r^2\, dr$ can be expressed as

$$W = -\sigma w/2\varepsilon^2 \tag{2.14}$$

where

$$w = \alpha\beta + \gamma, \qquad \beta = \varepsilon\mu/\sigma, \qquad \gamma = \int_0^\infty y^2 z^2 x^2\, dx \tag{2.15}$$

It is most straightforward to integrate the equations from $x = 0$ by prescribing either $y(0) = a$ or $z(0) = b$ and solving for the other in such a manner as to make $y \to 0$ exponentially as $x \to \infty$. It then follows that z approaches $\alpha/x + \beta$ asymptotically, as required in the Coulomb limit, with $\beta^2 < 1$, and one can thereby determine a and β. With the solution known, integrations determine α (again) and γ and hence the energy of the particle (in addition to the charge, found from α) for a given parameter pair σ and ε which appear in the Lagrangian.

To resolve the question of choice of solution, it was noted (Rosen, 1939; Menius and Rosen, 1942) that a local minimum for the energy exists in the neighborhood of the solution determined by the parameters $a = 1.63$, $b = 2.21493$, but the energy itself was negative, with the value of w approximately 2.83.

This system of equations has now been studied more thoroughly. It was found that for values of a up to approximately 3.55, particlelike solutions exist with $w \lesssim 2.83$. A gap then occurs in the range $3.55 \lesssim a \lesssim 3.62$, where $\beta^2 > 1$ and the solutions are oscillatory and hence do not represent particles. In the range $3.62 \lesssim a \lesssim 8.4$, a new set of particle states is found with $12.7 \lesssim w \lesssim 13.9$. With further increase of a, a second set of oscillatory solutions occurs, followed by a third regime of particle states with $w \sim 32$. Apparently this process continues indefinitely. However, there is no indication that any solution exists where the electromagnetic part of the energy density ever dominates over the "matter" part to render w negative and hence the total energy W positive.

The case of a scalar field without any electromagnetism was treated with the help of the Lagrangian (Rosen and Rosenstock, 1952)

$$L = (\partial^\mu\psi)(\partial_\mu\bar{\psi}) - \sigma^2\psi\bar{\psi} + (1/2)g\psi^2\bar{\psi}^2 \qquad (g > 0) \tag{2.16}$$

This differs from that of (2.1) (with $A^\mu = 0$) in that a quartic term has been added, and the signs of the first two terms have been changed. In the static, spherically symmetric case, with $\psi = \theta(r)e^{i\omega t}$, the field equation obtained from (2.16) has the form

$$d^2\theta/dr^2 + (2/r)\,d\theta/dr - \alpha^2\theta + g\theta^3 = 0 \tag{2.17}$$

and it has particlelike solutions of the asymptotic form $\theta = Ae^{-\alpha r}/(g)^{1/2}r$, where $A = \text{const}$ and $\alpha^2 = \sigma^2 - \omega^2 > 0$. The energy density in this case is found to be

$$T_{00} = \omega^2\theta^2 + \theta'^2 - (1/2)g\theta^4 \qquad ('=d/dr) \tag{2.18}$$

which is again indefinite for $g > 0$. However, an integration by parts making use of (2.17) yields a positive-definite total energy

$$W = 4\pi \int_0^\infty [2\omega^2\theta^2 + (1/2)g\theta^4]r^2\,dr \tag{2.19}$$

The particlelike solutions are quantized, and the ground-state (nodeless) solution, as well as the first-excited-state (one node) solution, were plotted in Rosen and Rosenstock (1952).

3. LEPTON PROPERTIES AND MODEL CONSIDERATIONS

At this point, we briefly review the properties of leptons. The known leptons, apart from the (supposedly) massless neutrinos, are the electron (e), the muon (μ), and the tau (τ). The electron is a stable (mean lifetime $>5\times10^{21}$ years), spin-1/2 fermion with mass m_e of 0.511 MeV. In geometrical units, which we shall use in this paper, the electron mass is 6.77×10^{-56} cm, and the charge is 1.38×10^{-34} cm. It does not exhibit structure at the current limit of resolution, $\sim10^{-16}$ cm (Quigg, 1983). The muon, which is also a spin-1/2 fermion, has a mass of 206.77 m_e and a mean lifetime of 2.197×10^{-6} sec. Its primary decay mode (98.6%) is into $e^- + \bar{\nu}_e + \nu_\mu$ (Okun, 1982). Both the electron and muon are observed directly, whereas the more recently discovered tau is deduced through its decay products. The tau has a mass of approximately 3491.6 m_e and is also a spin-1/2 fermion. It has decay modes into muon and neutrinos, electron and neutrinos, and hadrons plus neutrals (Okun, 1982). The g-factors of the electron and muon are very close to 2 as expected for Dirac particles, and this is sometimes cited as support for their identification as "point" particles (Quigg, 1983).

A "point," which is a singularity, is incompatible with a proper field theory. Consequently, we begin with the assumption that the leptons which are to be described are constructed from fields in a nonlinear manner and have finite size. In what follows, we will show that lepton structure does not negate the notion of their being "elementary" particles. From high-energy scattering experiments, it appears that the upper limit to the particle size is $\sim 10^{-16}$ cm. We arrive at a lower limit from a consideration of gravitation. For any given positive mass, it would be expected that there would exist a size for the corresponding particle at which level gravitation would play an important role in its structure. Assuming that the electron is a static, spherically-symmetric structure, the authors have found that in every attempted model constructed for it in which gravitation plays a significant role, the ratio e/m found for it was of the order unity or less. Moreover, as gravitation became a dominant force with the gravitational field gradients approaching enormous magnitudes, the ratio e/m approached 1 in every case. We suggest that this property may have great generality. The metric is approaching that of the "critically-charged" (Cooperstock and De La Cruz, 1979) Reissner–Nordström particle, with metric

$$\begin{aligned} ds^2 &= (1-m/r)^2\, dt^2 - (1-m/r)^{-2}\, dr^2 - r^2\, d\Omega^2 \\ d\Omega^2 &\equiv d\Theta^2 + \sin^2\Theta\, d\Phi^2 \end{aligned} \tag{3.1}$$

and $m^2 = e^2$. However, e/m is approximately 2.04×10^{21} for the electron, which is very far from unity. This suggests that elementary particles such as the electron are not concentrated to such a small dimension that gravitation plays an important role. Typically, the radius of a particle with the known mass and charge of an electron would be $\sim 10^{-33}$ cm at the stage at which gravitation would play an important role [see (3.2) below]. But if it is a general property that gravitation plays an important role only for $e^2 \sim m^2$ elementary particles, then we deduce a lower limit for the electron radius at a value appreciably above 10^{-33} cm.

The metric (3.1) has interesting properties. It stands at the junction between those Reissner–Nordström metrics which are said to exhibit "naked singularities" and those supposedly "clothed with a horizon." It is also the metric which is approached by the charged Curzon particle as it sheds its higher multipoles in the limit as it approaches the critically charged state (Cooperstock and De La Cruz, 1979).

Assuming that the electron, however small, is not a point, we can deduce other interesting aspects of its composition and character from the associated metric, in spite of the fact that gravitation plays a negligible role in its structure. From the 0–0 component of the external metric, as given by the Reissner–Nordström solution

$$g_{00} = 1 - 2m_e/r + e^2/r^2 = 1 - 2(m_e - e^2/2r)r \tag{3.2}$$

the mass that is concentrated within a sphere of radius r is (Cooperstock and De La Cruz, 1978)

$$m_{\text{eff}} = m_e - e^2/2r \tag{3.3}$$

Even at the largest possible electron radius of $\sim 10^{-16}$ cm, m_{eff} is negative, reaching zero at $r \sim 10^{-13}$ cm, from where it builds up its net positive value, achieving 99% of its total value m_e at $r \sim 10^{-11}$ cm. Thus, a good electron model should exhibit a negative-energy core. From the gravitational point of view, an electron would exhibit the phenomenon of Reissner–Nordström repulsion (Papapetrou, 1974) in the electrovacuum region between its core outer boundary and $r \sim 10^{-13}$ cm. Thus, we have an actual physical example of this unusual effect embodied in the electron. However, the gravitational force is too weak to be of importance for particles which we know of in nature that could be used as probes at present.

To build an electron, the simplest model which suggests itself is to add charge to the Lagrangian of Rosen and Rosenstock (1952) [equation (2.16)]. The original uncharged particle formed from (2.16) has a negative-energy core and a positive total mass [equation (2.19)]. Moreover, its field equation (2.17) yields quantized particle solutions for successive excitation states. It is natural to envision that quantized excitation states, which occur in the atomic and nuclear domains, should carry over to the level of fundamental particles. Indeed, experimental observations of the decays of very short-lived resonance states and decays of long-lived particles such as muons lend support to such a picture. Also, this provides a pathway to the construction of the observed particles in nature from fewer fundamental constituents, which has been the perennial goal of theoretical physics. Thus, in the case of leptons, the ground state is seen as the stable electron, with the muon, tau, etc., being represented by excited states. Decays from excited levels to the electron ground state with the emission of (supposedly) zero-rest-mass neutrinos would fit in well with analogous processes in atomic and nuclear physics.

The ground state for this model of charge added to (2.16) was investigated by numerical integration. The largest σ, which is the reciprocal of the scale length (and hence of the order of the particle size), was found to be of order $10^{12}\ \text{cm}^{-1}$. However, experimentally, a minimum σ of the order of $10^{15}\ \text{cm}^{-1}$ is required for the electron. Thus, this simplest model yields an electron which is too large; its radius is of the order of the classical electron radius.

At the next level of complexity, we considered a field theory incorporating both the original scalar ψ with electromagnetism of (2.1), which gave a negative-energy particle, coupled to the scalar of (2.16) (now called ψ_1), previously unassociated with electromagnetism, and having positive energy. The simplest form of coupling to give a gauge-invariant contribution to the

Lagrangian is $f\psi\bar{\psi}\psi_1\bar{\psi}_1$ where f is a coupling constant. The Lagrangian thus has the form

$$L = -F_{\mu\nu}F^{\mu\nu}/8\pi + \partial^{\mu}\psi_1\partial_{\mu}\bar{\psi}_1 - \sigma^2\psi_1\bar{\psi}_1 + (1/2)g\psi_1^2\bar{\psi}_1^2 - (D^{\mu}\psi)\overline{(D_{\mu}\psi)} + \sigma^2\psi\bar{\psi} - f\psi\bar{\psi}\psi_1\bar{\psi}_1 \tag{3.4}$$

With the proper choice of parameters, it is possible to create an electron model with σ's even well in excess of 10^{15} cm^{-1} if desired, and hence of minute dimensions. However, if the goal is more ambitious, namely to have the next excited state of ψ_1 represent the muon with the same set of parameters, then it is found that over a wide range of coupling strengths, σ is consistently in the range $(2\text{–}4)\times 10^{14}$ cm^{-1}, which is too small. Furthermore, regardless of the coupling, no single set of parameters could be found which also encompasses the second excited state of ψ_1 with the correct mass to represent the tau.

However, this model has two attractive features which will be useful in what follows. First, the coupling between the scalars induces a binding-energy contribution in the energy integral which grows in absolute magnitude with excitations. This presents us with a mechanism to bind the particle constituents together, regardless of the level of excitation. Indeed, should it be possible to extend the methods discussed in this paper to describe the formation of hadrons from quarks, such a mechanism could possibly realize the phenomenon of quark confinement. In the present context, the mechanism serves an analogous function in that there is no evidence to suggest that an electron could ever be decomposed. Moreover, it would be disturbing if it could be decomposed within the present framework, since one of the freed constituents would have negative mass. Thus, we have the coexistence of the concepts of "elementarity" and structure with no contradiction.

Second, the model is attractive in that the negative-energy scalar carries the charge, while excitations of the positive-energy scalar, for reasonable values of the coupling constant, affect the charge only slightly. Hence, it is easy to readjust the parameters of the negative-energy scalar so as to maintain charge conservation for all the states.

In the following section, we describe in some detail a more satisfactory model for leptons which incorporates these attractive features.

4. SUCCESSFUL MODELS OF LEPTONS

In Section 3, we saw that the use of two scalar fields, one neutral and one charged, had some attractive features, but it failed to give a satisfactory theory of leptons. The addition of a third, neutral, scalar field enables us to reach this goal. Perhaps future studies will reveal that three fields underlie the structure of leptons, which are spin-1/2 fermions, in the same sense that three quarks underlie the structure of baryons, which are also spin-1/2 fermions.

Thus, we introduce the Lagrangian

$$\begin{aligned} L = & -F_{\mu\nu}F^{\mu\nu}/8\pi + \partial^\mu\psi_1\partial_\mu\bar{\psi}_1 - \sigma^2\psi_1\bar{\psi}_1 + (1/2)g_1\psi_1^2\bar{\psi}_1^2 + \partial^\mu\psi_2\partial_\mu\bar{\psi}_2 \\ & -\sigma^2\psi_2\bar{\psi}_2 + (1/2)g_2\psi_2^2\bar{\psi}_2^2 - (D^\mu\psi)\overline{(D_\mu\psi)} + \sigma^2\psi\bar{\psi} \\ & -f_1\psi\bar{\psi}\psi_1\bar{\psi}_1 - f_2\psi\bar{\psi}\psi_2\bar{\psi}_2 \end{aligned} \tag{4.1}$$

As before, ψ is the complex scalar of the form in (2.1), which is directly coupled to the electromagnetic field, ψ_1 and ψ_2 are scalars of the positive-energy form of (2.16), and each of which is coupled to the negative-energy scalar (coupling constants f_1 and f_2). An additional coupling term between ψ_1 and ψ_2 could have been added. This was not done, partly because a successful model is achieved without it, and partly because at this early stage in the development of the theory, there is no clear directive to guide such an inclusion.

Variations with respect to $\bar{\psi}$, $\bar{\psi}_1$, $\bar{\psi}_2$, and A^μ, respectively, yield the set of nonlinear coupled field equations

$$(D^\nu\psi)_{;\nu} - i\varepsilon A_\nu(D^\nu\psi) + \sigma^2\psi - f_1\psi\psi_1\bar{\psi}_1 - f_2\psi\psi_2\bar{\psi}_2 = 0 \tag{4.2}$$

$$(D^\nu\psi_1)_{;\nu} + \sigma^2\psi_1 - g_1\psi_1^2\bar{\psi}_1 + f_1\psi\bar{\psi}\psi_1 = 0 \tag{4.3}$$

$$(D^\nu\psi_2)_{;\nu} + \sigma^2\psi_2 - g_2\psi_2^2\bar{\psi}_2 + f_2\psi\bar{\psi}\psi_2 = 0 \tag{4.4}$$

$$F^{\mu\nu}_{;\nu} = 4\pi J^\mu, \qquad J^\mu \equiv -(1/2)i\varepsilon(\bar{\psi}D^\mu\psi - \psi\overline{D^\mu\psi}) \tag{4.5}$$

Particularizing to static spherical symmetry as before, we take ψ_1 and ψ_2 to be real; set

$$\begin{aligned} A^\mu = (\varphi(r), 0, 0, 0), \qquad \psi = \theta(r)e^{-i\varepsilon\mu t} \\ \psi_1 = \theta_1(r), \qquad \psi_2 = \theta_2(r) \end{aligned} \tag{4.6}$$

and express the field equations as

$$\nabla^2\theta + \varepsilon^2\theta(\varphi+\mu)^2 - \sigma^2\theta + f_1\theta\theta_1^2 + f_2\theta\theta_2^2 = 0 \tag{4.7}$$

$$\nabla^2\theta_1 - \sigma^2\theta_1 + g_1\theta_1^3 - f_1\theta^2\theta_1 = 0 \tag{4.8}$$

$$\nabla^2\theta_2 - \sigma^2\theta_2 + g_2\theta_2^3 - f_2\theta^2\theta_2 = 0 \tag{4.9}$$

$$\nabla^2\varphi + 4\pi\varepsilon^2(\varphi+\mu)\theta^2 = 0 \tag{4.10}$$

For the purpose of numerical integration, it is useful to work with the dimensionless variables

$$\begin{aligned} x = r\sigma, \qquad z = (\varepsilon/\sigma)(\varphi+\mu), \qquad y = \varepsilon(4\pi)^{1/2}\theta/\sigma \\ y_1 = g_1^{1/2}\theta_1/\sigma, \qquad y_2 = g_2^{1/2}\theta_2/\sigma \end{aligned} \tag{4.11}$$

in terms of which the field equations are ($' = d/dx$)

$$y'' + (2/x)y' + z^2 y - y + (f_1/g_1)yy_1^2 + (f_2/g_2)yy_2^2 = 0 \qquad (4.12)$$

$$y_1'' + (2/x)y_1' - y_1 + y_1^3 - (f_1/4\pi\varepsilon^2)y^2 y_1 = 0 \qquad (4.13)$$

$$y_2'' + (2/x)y_2' - y_2 + y_2^3 - (f_2/4\pi\varepsilon^2)y^2 y_2 = 0 \qquad (4.14)$$

$$z'' + (2/x)z' + y^2 z = 0 \qquad (4.15)$$

The charge and mass are

$$e = \alpha/\varepsilon \qquad (4.16)$$

$$m = \sigma w/2\varepsilon^2 \qquad (4.17)$$

where

$$w = (2\pi\varepsilon^2/g_1)\int_0^\infty y_1^4 x^2\,dx + (2\pi\varepsilon^2/g_2)\int_0^\infty y_2^4 x^2\,dx$$

$$-(f_1/g_1)\int_0^\infty y^2 y_1^2 x^2\,dx - (f_2/g_2)\int_0^\infty y^2 y_2^2 x^2\,dx - \alpha\beta - \gamma$$

with α, β, and γ having the same meanings as in Section 2.

The first formulation which might come to mind in constructing the family of leptons is modeled upon a structure considered in Section 3. This is to leave both ψ and one of ψ_1 and ψ_2 always in the ground state and represent the electron, muon, and tau, respectively, by the ground, first excited, and second excited states of the second positive-energy scalar. However, as before, no choice of interaction could be found which could give all three known particles.

Two other formulations, however, are successful. One is to represent the electron as the ground state of all three scalars (ψ, ψ_1, ψ_2), the muon by the first excitation (*) of ψ_1, i.e., (ψ, ψ_1^*, ψ_2), and the tau by the first excitation of ψ_2, i.e., (ψ, ψ_1, ψ_2^*). Such a formulation will lead to the correct masses for e, μ, and τ. However, the decay modes of τ to both μ and e are equally likely (Okun, 1982; Anon, 1986), whereas, with the deexcitation of the tau as expressed above, we are only able to achieve the electron state in a natural manner. (Nevertheless, we end this section with a data set for this model, should it turn out to be the correct one.)

An alternative formulation avoids this difficulty. We consider the electron, muon, and tau given by (ψ, ψ_1, ψ_2), (ψ, ψ_1^*, ψ_2), and (ψ, ψ_1^*, ψ_2^*), respectively. A deexcitation of ψ_2 alone takes τ into μ, a deexcitation of ψ_1 then takes μ into e, while a deexcitation of (ψ_1^*, ψ_2^*) takes τ into e. In this formulation, we would postulate that in high-energy collisions the muon is created by the excitation of ψ_1, while very high-energy collisions which suffice to excite the ψ_2 generally also excite the more weakly coupled ψ_1; hence, the representation of the tau by (ψ, ψ_1^*, ψ_2^*). In the process of deexcitation, however, we could have an intermediate state (ψ, ψ_1, ψ_2^*) (with energy 3288 m_e for the parameter set below), which could be related to the decay mode of tau into hadrons and neutrals.

It must be stressed that the solutions found are static, and hence no predictions can be made for transition probabilities, which are related to dynamics. Even if such solutions could be found, it is far from clear that these could successfully replicate the observed dynamical processes in nature, because the theory at the present level is not a quantum field theory. However, the classical theory is the first step in any event. Also, the history of physics has various examples in which quantized reformulations of classically constructed theories yield little or no change.

The results of numerical integration for the parameter set

$$f_1/4\pi\varepsilon^2 = 0.0005, \qquad f_2/4\pi\varepsilon^2 = 0.003$$
$$f_1/g_1 = 2.7709\times 10^{-5}, \qquad f_2/g_2 = 2.6528\times 10^{-3} \tag{4.18}$$

(with the first two chosen for convenience and the last two then taken so as to give the correct mass ratios) are given in Table I. Here, a, b, c, and d are, respectively $y(0)$, $z(0)$, $y_1(0)$, and $y_2(0)$; the last column represents the ratio of the w's for the particle states relative to the w of the electron ground state (w_e). From (4.17), fixed parameters σ and ε together with the ratios of the w's yield the ratios of the corresponding masses. We note that the muon and tau masses found are only 0.06% removed from their observed (Anon, 1986) values of 206.77 m_e and (approximately) 3492 m_e (± 6 m_e) for the μ and τ.

From equations (4.16) and (4.17),

$$\sigma = 2\alpha^2 m_e/w_e e^2 \tag{4.19}$$

and from the known values of α and w of the ground state (see Table I) and the known values of e and m for the electron, we find that σ, the reciprocal of the scale size, is 6.013×10^{15} cm^{-1}. It should be noted that a value of σ corresponds to a particle size of the order of $1/\sigma$ ($=1.663\times 10^{-16}$ cm in the present case). It is interesting that the value of $1/\sigma$ agrees with the experimental upper limit for the lepton size and that it is obtained

Table I. Lepton Parameter Values

P	a	b	c	d	α	β	γ	w_i	w_i/w_e
e	1.6300	2.2112	4.3383	4.3431	1.9051	0.00264	2.8168	4.286×10^{-3}	1
μ	1.6300	2.2112	14.1038	4.3431	1.9051	0.00264	2.8168	0.8869	206.9
τ	1.6243	2.2025	14.1038	14.1049	1.9051	0.00489	2.8077	14.974	3494

simply by fitting the parameters to the known masses (and the charge) of the leptons. From equation (4.16), $\varepsilon = 1.379\times 10^{34}\,\mathrm{cm}^{-1}$ and from (4.18), $f_1 = 1.1948\times 10^{66}\,\mathrm{cm}^{-2}$, $f_2 = 7.1690\times 10^{66}\,\mathrm{cm}^{-2}$, $g_1 = 4.3120\times 10^{70}\,\mathrm{cm}^{-2}$ and $g_2 = 2.7024\times 10^{69}\,\mathrm{cm}^{-2}$.

Until we know the actual values for the coupling parameters f_1 and f_2 or, alternatively, the intrinsic binding parameters g_1 and g_2, we cannot predict with precision the other excitation states. In the above solution we simply postulated the values of $f_1/4\pi\varepsilon^2$ and $f_2/4\pi\varepsilon^2$, taking the second value to be larger than the first. This corresponds to stronger binding for the ψ_2 scalar relative to ψ_1 as the excitation of the former creates the tau. With this parameter set, there are states of higher excitation

$$(\psi, \psi_1^{**}, \psi_2), \quad (\psi, \psi_1^{***}, \psi_2), \quad (\psi, \psi_1^{*(4)}, \psi_2), \quad (\psi, \psi_1^{**}, \psi_2^{*})$$
$$(\psi, \psi_1^{***}, \psi_2^{*}), \quad (\psi, \psi_1^{*(5)}, \psi_2), \quad (\psi, \psi_1^{*(4)}, \psi_2^{*}), \quad \text{etc.}$$

at the respective energies 703, 1611, 3044, 3990, 4899, 5124 and 6331 times m_e, etc. Presumably, they would be very short-lived. It would be interesting to determine whether there is any experimental evidence in heavy-lepton searches for short-lived states in the neighborhoods of these energies.

We now return to the model in which the e, μ, and τ are represented, respectively, by the states (ψ, ψ_1, ψ_2), $(\psi, \psi_1^{*}, \psi_2)$, and $(\psi, \psi_1, \psi_2^{*})$. We begin with the same $f_1/4\pi\varepsilon^2 = 0.0005$ and $f_2/4\pi\varepsilon^2 = 0.003$ as in (4.18), but now determine new parameter values $f_1/g_1 = 2.6160\times 10^{-5}$ and $f_2/g_2 = 2.6618\times 10^{-3}$ appropriate to forming the leptons with the above structure of excitations. With this parameter set, the states of higher excitation

$$(\psi, \psi_1^{**}, \psi_2), \quad (\psi, \psi_1^{***}, \psi_2), \quad (\psi, \psi_1^{*(4)}, \psi_2), \quad (\psi, \psi_1^{*}, \psi_2^{*})$$
$$(\psi, \psi_1^{**}, \psi_2^{*}), \quad (\psi, \psi_1^{***}, \psi_2^{*}), \quad (\psi, \psi_1^{*(5)}, \psi_2), \quad (\psi, \psi_1^{*(4)}, \psi_2^{*}), \quad \text{etc.}$$

occur at the respective energies 702, 1607, 3037, 3691, 4187, 5092, 5111, and 6522 times m_e, etc. Note that what was formerly the tau is now a state of higher excitation and that the former intermediate state $(\psi, \psi_1, \psi_2^{*})$ has now assumed the role of the tau.

Returning to equation (4.17), we now consider the changes which occur as one goes to successively higher excitation states of y_1 and y_2. Since α, β, and γ are related to the negative-energy scalar y, which does not get excited, there is little change in these quantities. Indeed, insofar as α is concerned, the start value, a, for y is adjusted so that α is conserved. This has the effect of maintaining charge conservation [see (4.16)] for successive lepton states. Since excitation increases y_1 and y_2, the essential effect is to increase m through the increase in the first two integrals and, to a less pronounced degree, decrease m through the increasing binding given by the third and fourth integrals. Thus, successive excitations have the effect of increasing the binding so that the composite structure is never decomposed.

An analogous mechanism is conceivable in the case of quark confinement in hadrons. However, one basic difference is that the quarks have separated centers and hence the mass integrals could present new features.

5. THE ROLE OF GRAVITATION

We now consider the effect of gravitation in charged-particle formation. This was done for a variety of particle models considered by the present authors. However, for simplicity of exposition, we focus on the simplest of these models.

In Rosen (1939), in the absence of gravitation, the sign structure of the Lagrangian was chosen as in (2.1) because only this form led to field equations for which there existed particle solutions, albeit with negative mass. Positive-mass particles with no charge were found in Rosen and Rosenstock (1952) with the opposite sign structure in the Lagrangian (2.16) by the addition of a binding quartic g term. It is natural to inquire whether the Lagrangian of (2.1) with charge, modified in sign structure to that of (2.16) so as to make the energy positive, could yield particle solutions, given sufficient gravitational binding. If so, this charged-particle model would have gravitation play the role of the g term in (2.16).

Thus, we consider the Lagrangian

$$L = -F_{\mu\nu}F^{\mu\nu}/8\pi + (D_\mu\psi)(\overline{D^\mu\psi}) - \sigma^2\psi\bar{\psi} \tag{5.1}$$

where now the metric is for static, spherically symmetric curved space-time

$$\begin{gathered} ds^2 = e^\nu\,dt^2 - e^\lambda\,dr^2 - r^2\,d\Omega^2 \\ \lambda = \lambda(r), \qquad \nu = \nu(r) \end{gathered} \tag{5.2}$$

After the variations are performed and the coordinates are scaled as in (2.10), the field equations $(' = d/dx)$

$$d^2(xy)/ds^2 = xy(1 - z^2e^{-\nu})e^\lambda + (xy'/2)(\lambda' - \nu') \tag{5.3}$$

$$d^2(xz)/dx^2 = xy^2ze^\lambda + (xz'/2)(\lambda' + \nu') \tag{5.4}$$

are derived. These are the analogues of equations (2.11). [Note that with $\lambda = \nu = 0$, only (5.4) is seen to be different in sign relative to (2.11) because it is only in the variation with respect to A^μ that the sign differences between (5.1) and (2.1) can play a role.]

Two additional equations are required for the new functions λ and ν, and these were obtained from the Einstein gravitational field equations involving the T_0^0 and T_r^r components of the energy-momentum tensor:

$$\lambda' = (1 - e^\lambda)/x + (\sigma^2 x/\varepsilon^2)(y^2 z^2 e^{\lambda-\nu} + y'^2 + y^2 e^\lambda + z'^2 e^{-\nu}) \qquad (5.5)$$

$$\nu' = (e^\lambda - 1)/x + (\sigma^2 x/\varepsilon^2)(y^2 z^2 e^{\lambda-\nu} + y'^2 - y^2 e^\lambda - z'^2 e^{-\nu}) \qquad (5.6)$$

Equations (5.3)-(5.6) were integrated numerically with various initial values for $y(0) = a$ and $z(0) = b$, where λ, ν, λ', ν', y', z' were set equal to zero at $x = 0$. [It should be noted that $\nu(0)$ was taken to vanish for convenience.This means that $\nu(\infty) \neq 0$, so that we have a rescaling of time.] The parameter ratio σ^2/ε^2 was then adjusted to minimize y^2 as the exterior of the particle is approached. This has the effect of localizing the particle sharply, in analogy to the localization of the probability density $\Psi\bar{\Psi}$ in quantum mechanics. As expected, the improvement in the localization is correlated with the increasing convergence to the Reissner-Nordström metric outside of the particle, i.e., in the direction of perfect electrovacuum. It is also understandable physically that, as was found, this trend is accompanied by increasing gravitational field intensity: strong gravitational fields are more effective in concentrating the particle.

Data for a model with $a = 3$, $b = 40$ are now described. The maximum concentration was found for $\sigma^2/\varepsilon^2 = 0.538268624309411$. The most intense gravitational field gradients $(\lambda'_M, \nu'_M) = (-2.16 \times 10^5, 3.54 \times 10^5)$ occurred at $x = 1.54614$ and the maximum electric field gradient $z'_M = 2.07 \times 10^{13}$ occurred nearby at $x = 1.54628$. At $x = 1.55130$, one is exterior to the particle since $\nu' = -\lambda'$ $(=3.86271 \times 10^2)$, as expected for the Reissner-Nordström metric [see (5.7) below] for arbitrary scaling of the time coordinate. At $x \simeq 1.546$, $y^2 \simeq 10^{-4}$, whereas at $x \simeq 1.6$, $y^2 \simeq 10^{-8}$, and by $x \simeq 3$, $y^2 \simeq 10^{-10}$. Thus, the particle is highly concentrated. The choice of smaller values for σ^2/ε^2 leads to a more diffuse particle with smaller field gradients. Note that in this paper, we used geometrical units in which $c = G = 1$. Had we not taken $G = 1$, then the parameter σ^2/ε^2 in (5.5) and (5.6) would have appeared as $G\sigma^2/\varepsilon^2$, and it would have displayed its character more transparently as being related to the strength of the gravitational interaction.

For arbitrary e and m values, the Reissner-Nordström metric is (5.2) with

$$e^{-\lambda} = 1 - 2m/x + e^2/x^2 = \text{const} \cdot e^\nu \qquad (5.7)$$

using dimensionless units. With the data values $\lambda = 3.45798$ and $\lambda' = -3.87655 \times 10^{-2}$ at $x = 9.737814$, we find, using (5.7), that $e = 1.546163$ and $m = 1.546148$. The near equality of these values shows that the metric is very close to the critically charged form (3.1) (now with r replaced by x), and it is clear that the huge field gradients, in turn, should occur in the vicinity of $x = m$, as above.

It is of considerable interest that in all the cases studied of this model and in all the cases studied in other models, such as with the addition of a quartic g term, with other coupled scalars, etc., the same behavior was witnessed: the critically charged Reissner–Nordstöm solution was approached as gravity was made to assume a dominant role.

In other situations where gravity played a subsidiary but still detectable role, e/m was always found to be of the order unity or less. When e/m was taken to have a large value, as, for example, in the case of an electron, it was always found that the effect of gravitation was negligible. Thus, the evidence suggests that gravitation does not play a significant role in the structure of the elementary particles with which we are familiar.

6. DISCUSSION

The idea of describing an elementary particle as a singularity-free concentration of fields is very attractive. One wishes to avoid a singularity since, at a singular point, the field equations break down, and the physical laws which they describe do not hold. However, the structure of such a particle presents a challenge: how can one have a particle in which the various volume elements attract one another and yet have a net positive mass? One answer was provided earlier: the particle should consist of a core of negative mass surrounded by a region with sufficient positive mass. There may be other means of achieving this end, but at least in the case of the electron, as discussed in Section 3, this is a necessity. In any event, the fields should provide both attraction and repulsion, since with only one of these present, the particle will either collapse or explode.

One can think of gravitation as providing the required attractive force between positive-mass volume elements. However, as was pointed out in Section 3, gravitation cannot play an important role in the structure of the electron in view of the large e/m ratio ($\sim 10^{21}$). In the case of the electron, the situation is aggravated by the fact that the radius must be small ($\lesssim 10^{-16}$ cm). Moreover, if one regards the electron as one member of the lepton family, and one wishes to incorporate the other leptons, μ and τ, as well in the same framework, one is faced with stringent requirements. In the present work, these are satisfied by a model which involves the presence of three scalar fields, one charged and two neutral. In particular, it gives an electron size of the order of 10^{-16} cm, which is of the order of the present experimental upper limit. It is conceivable that this is the actual size of the electron and not just the upper limit.

It was stressed earlier that if one assigns a significant role to gravitation in the structure of the charged particle, one is led to a situation in which the charge is of the order of magnitude of, or less than, the mass. This has no relation to the elementary charged particles, having very large e/m ratios, with which we are familiar. However, it is still conceivable that some of the particles which we now regard as "elementary" are actually composed of more fundamental massive particles with nearly equal charge and mass, which do involve gravitation in their structure.

The stability analysis for our rather complicated Lagrangian is currently in progress and will be the subject of a future paper.

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